

The Born Rule and Time-Reversal Symmetry of Quantum Equations of Motion

Aleksey V. Ilyin*

Moscow Institute of Physics and Technology

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It was repeatedly underlined in literature that quantum mechanics cannot be considered a closed theory if the Born Rule is postulated rather than derived from the first principles. In this work the Born Rule is derived from the time-reversal symmetry of quantum equations of motion. The derivation is based on a simple functional equation that takes into account properties of probability, as well as the linearity and time-reversal symmetry of quantum equations of motion. The derivation presented in this work also allows to determine certain limits to applicability of the Born Rule.

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I. INTRODUCTION

The Born Rule, being one of the basic principles of quantum physics, establishes a link between a solution of wave equation and the results of observations. The Born Rule was proposed in 1926 on the grounds that it complies with the conservation of the number of particles in scattering process [1]. Later, the Born Rule was supplemented by the von Neumann projection postulate [2] to describe the results of repeated measurements.

From the beginning, the Born Rule was perceived as an axiom independent of quantum equations of motion. There were numerous attempts to derive the Born Rule from the basic quantum principles. For example, a derivation of the Born Rule was discussed in [3] and later in [4–6] within the many-worlds interpretation of quantum mechanics. However, this approach was criticized in [7]. The Born Rule was obtained in [8] within the concept of hidden measurements, which, however, is not a widely accepted interpretation of quantum mechanics. Recently, the Born Rule was derived in [9] from the unitarity of quantum evolution and an additional assumption about probability conservation.

In this paper, the Born Rule is derived from the basic quantum principles, namely, time-reversal symmetry and linearity of quantum equations of motion. The derivation presented here is based on the following premisses:

(α) The state of a particle is completely determined by its wave function (WF), which is a solution of quantum equation of motion, for example, the Schroedinger equation;

(β) It follows from the above principle that all properties of the particle, including the probability density P , shall be determined solely by the wave function ψ of the particle, i. e. $P = P(\psi)$.

Our purpose is to find function $P(\psi)$.

II. DERIVATION OF THE BORN RULE

Probability density $P(\psi)$ is assumed to be a *universal* function that is applicable to any state of quantum particle described by the WF. Therefore, it will be sufficient to find $P(\psi)$ in some simple special case, for example, in the case of a plane wave, while the solution found in this special case, due to its universality, is sure to be valid for an arbitrary WF. By probability density of finding a particle in a plane wave we mean the probability that the particle is in a unit volume.

A. Amplitudes

Consider a plane wave with complex amplitude S propagating along x -axis:

$$\psi(x, t) = S \exp i(kx - wt), \quad (1)$$

where k and w are proportional, respectively, to particle momentum and energy $k = \frac{p}{\hbar}$, $w = \frac{E}{\hbar}$.

Let a plane wave (1) be incident on a potential barrier characterized by complex reflectance r and transmittance t (labeling transmittance and time by the same symbol is traditional and may cause no confusion.) Then the reflected wave and transmitted wave (Fig. 1A) will have complex amplitudes, respectively,

$$a = Sr \quad \text{and} \quad b = St. \quad (2)$$

It is well known [10] that the Schroedinger equation is symmetric with respect to time reversal if, along with the substitution $t \rightarrow -t$, the WF is changed to its complex conjugate (see Appendix for a simple explanation).

Let the simultaneous action of time reversal and complex conjugation be denoted as \sim , i. e. $\tilde{\psi}(x, t) = \psi^*(x, -t)$. This operation yields a time-reversed state of quantum particle. In the general case, time-reversibility of the WF should be regarded as a formal mathematical property of quantum equations of motion. That property does not imply that any state of quantum particle can actually be reversed in an experiment.

* a.v.ilyin@mail.mipt.ru

Thus, reversing quantum state (1) yields

$$\tilde{\psi}(x, t) = S^* \exp i(-kx - \omega t). \quad (3)$$

Therefore, the reversed plane wave has a complex-conjugate amplitude S^* .

Now let us reverse the process of interaction of a particle with the barrier. Two waves with amplitudes a and b that were departing from the barrier now will go back to the barrier and their complex amplitudes will turn into conjugate amplitudes a^* and b^* respectively (Fig. 1B). Assuming the transmittance and reflectance of the potential barrier being the same in either direction (that is certainly true for a symmetrical barrier), one obtains the amplitude of the wave reflected to the left as ra^* and transmitted to the left as tb^* . When time is reversed, a wave of amplitude S^* should appear running away from the barrier (Fig. 1B.)

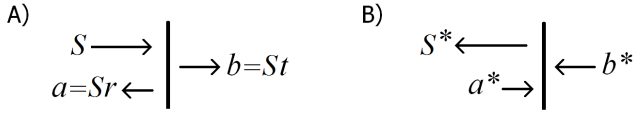


Figure 1. Direct (A) and time-reversed (B) processes of particle interaction with the barrier. Complex amplitudes a of reflected wave and b of transmitted wave become complex-conjugates a^* and b^* under time reversal (see explanation in the text.)

It follows from the linearity of quantum equations of motion that the superposition principle holds for the WF. This conclusion applies to both direct and time-reversed processes. Therefore, the reversed state S^* is the result of superposition of two waves running to the left from the barrier, namely, the reflected wave ra^* and transmitted wave tb^* :

$$ra^* + tb^* = S^*. \quad (4)$$

Exponential factors $\exp i(-kx - \omega t)$ of all waves cancel in this equation.

Multiplying (4) by S and taking into account (2) one obtains

$$aa^* + bb^* = SS^*, \quad (5)$$

which is an inevitable consequence of time-reversibility of quantum equations and the superposition principle.

B. Probabilities

A particle falling on the barrier is *indivisible*, so that it can be found with probability $P(a)$ in the reflected beam or with probability $P(b)$ in the transmitted beam (Fig. 1A). As these events are mutually exclusive (i.e. a particle cannot be found both in the reflected and transmitted beams) the probabilities should obey

$$P(a) + P(b) = P(S). \quad (6)$$

Probability $P(\psi)$ must be a real number that, according

to Provision (β), should be obtained from complex WF. Real x can be obtained from complex ψ using infinite number of ways. For example, one may take a real part of complex number $x = \text{Re}(\psi)$, or complex number argument $x = \text{Arg}(\psi)$, or complex number modulus $x = |\psi| = \sqrt{\psi\psi^*}$. Also, various combinations of above expressions will produce real numbers.

In order to select a feasible expression for real number $P(\psi)$ from among the multitude of expressions, the following physical principle should be applied: probability $P(\psi)$ should be independent of arbitrarily chosen time origin because the properties of infinite plane wave do not depend on time.

Only expression $\psi\psi^*$ will have this property while other expressions mentioned above will explicitly depend on time through the WF argument $kx - \omega t$. Therefore, with no loss of generality one may assume that $P(\psi)$ is some function of argument $x = \psi\psi^*$. In other words,

$$P(\psi) = F(\psi\psi^*). \quad (7)$$

Such a designation is justified because it imposes no additional restrictions on the unknown function $F(x)$. Therefore, equation (6) can be written as

$$F(aa^*) + F(bb^*) = F(SS^*). \quad (8)$$

According to Provision (β), the probability can be a function only of the WF amplitude, which is provided by equation (8).

C. The Born Rule

Due to (5) equation (8) can be written as

$$F(aa^*) + F(bb^*) = F(aa^* + bb^*). \quad (9)$$

That is a functional equation with respect to the unknown function $F(x)$. Equation (9) can be presented in the usual form:

$$F(x) + F(y) = F(x + y), \quad (10)$$

where $x = aa^*$, $y = bb^*$.

It is important to note that here x and y are independent variables because the height of the barrier (and its transparency) can be altered independently of amplitude S of the incident wave. Therefore, the only solution to functional equation (10) is a linear function

$$F(x) = kx = kaa^*, \quad (11)$$

which in view of (7) yields

$$P(a) = kaa^*. \quad (12)$$

Constant k is actually a normalization factor that defines the WF unit of measurement. In each quantum problem this normalization factor is chosen based on convenience. For example, in the problem considered here, it is convenient to admit that the unit amplitude $a = 1$ should

correspond to probability $P(a) = 1$, from which one obtains $k = 1$. Therefore, (12) yields the Born Rule

$$P(a) = aa^*, \quad (13)$$

i.e. the sought-for relationship between the probability density and the WF amplitude.

The particle state $|\psi\rangle$ after interaction with the barrier, according to the wave equation solution, is given by

$$|\psi\rangle = a|-\rangle + b|+\rangle, \quad (14)$$

where $|-\rangle$ and $|+\rangle$ are the states with negative and positive momenta, i.e. plane waves running to the left and to the right from the barrier, respectively. State vectors $|-\rangle$ and $|+\rangle$ are, actually, orthonormal eigen states of the momentum operator. Therefore, multiplying (14) by $\langle -|$ yields

$$\langle -|\psi\rangle = a. \quad (15)$$

Now from (13) and (15) one obtains for the probability of realization of particle state $|-\rangle$

$$P(a) = aa^* = |\langle -|\psi\rangle|^2, \quad (16)$$

which is another way to put down the Born Rule. Formula (16) implies that if a particle is in state $|\psi\rangle$ then the probability that the particle is found in state $|-\rangle$ is the squared modulus of complex number $\langle -|\psi\rangle$, which is a projection of the original state $|\psi\rangle$ on the final state $|-\rangle$. This important quantum postulate is derived here from the time-reversal symmetry of quantum equation of motion.

Thus, the Born Rule (13) or (16) has been obtained as a simple consequence of time-reversal symmetry of quantum equation of motion. In addition to this symmetry, an important role in the present derivation is played by the superposition principle [Eq. (4)], which follows from the equation of motion, and by the assumption about the existence of indivisible particles [Eq. (6)], which does *not* follow from the equation of motion.

The Born Rule thus obtained for a particular case of plane wave interaction with a potential barrier shall be valid also in the general case because the derivation presented here is based on quantum principles (α) and (β) implying that the probability density $P(\psi)$ is a *universal* function applicable to any state of a particle that can be described by a WF.

III. LIMITS OF APPLICABILITY OF THE BORN RULE

In contrast to the Schroedinger equation, the relativistic Klein-Gordon equation contains the energy operator squared. Therefore, time reversal does not change the Klein-Gordon equation. For this reason, under time reversal, the Klein-Gordon equation does not require transition to conjugate states (according to the logic set out in the Appendix). The Born Rule, therefore, may be applicable to some particular solutions of the Klein-Gordon equation

but it should be inapplicable to the general solution of this equation.

IV. CONCLUSIONS

The Born Rule is derived from fundamental quantum principles, namely, from the time-reversal symmetry and linearity of quantum equations of motion. To obtain the Born Rule it is also necessary to admit the existence of probability and discrete quantum particles, which is *not* a consequence of quantum equations of motion.

It should be noted that if the Klein-Gordon equation is taken as the equation of motion in the above derivation then time-reversal $t \rightarrow -t$ will not change the equation and, therefore, it will not require complex conjugation of the WF. For this reason, the Born Rule should be inapplicable to some solutions of the Klein-Gordon equation.

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APPENDIX

Let us consider the Schroedinger equation for an arbitrary state vector $|\psi\rangle$:

$$\hat{E}|\psi\rangle = \hat{H}|\psi\rangle, \quad (17)$$

where $\hat{E} = i\hbar \frac{\partial}{\partial t}$ - is the energy operator, \hat{H} - is time-independent Hamiltonian.

Upon substituting $t \rightarrow -t$ in (17) one obtains

$$-\hat{E}|\psi^-\rangle = \hat{H}|\psi^-\rangle, \quad (18)$$

where $|\psi^-\rangle = |\psi(\mathbf{r}, -t)\rangle$. Equation (18) does not coincide with the Schroedinger equation (17) due to the "minus" sign on its left side. In order to return to the correct equation of motion while retaining time-reversal, it is necessary to apply the operation of Hermitian conjugation to eq. (18):

$$\langle \psi^- | \hat{E} = \langle \psi^- | \hat{H}, \quad (19)$$

where it is taken into account that the Hamiltonian is a Hermitian operator $\hat{H}^+ = \hat{H}$ while the energy operator is anti-Hermitian $\hat{E}^+ = -\hat{E}$ because $i\hbar \frac{\partial}{\partial t}$ changes its sign under complex conjugation.

Thus, if we demand that the reversed solution should satisfy the Schroedinger equation then time reversal of a quantum state should result in simultaneous replacement of each ket $|\psi\rangle$ with corresponding bra $\langle \psi^-|$. This operation applied to plane wave (1) results in complex conjugation of amplitude $S \rightarrow S^*$, which is taken into account in eq. (3).

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